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SOLUTION OF EQUATIONS OF MOTION FOR THE START OF AN ELECTRIC HOIST

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Summary: This contribution describes a solution of equations of motion for the start of an electric hoist. For the calculation there are used values for the actual electric hoist, thereby it is possible to verify an accuracy of the calculation and system sensitivity as well. In the second half of this paper, there are stated the first results for the start of an electric hoist with three degrees of freedom.

Key words: equation of motion, solution, electric hoist

INTRODUCTION

In former issue of this journal, concretely in (1), there were obtained the equations of motion for an electric hoist with three degrees of freedom (DOF) describing its start. This obtained system of three non-linear, second-order differential equations has not been solved in former issue due to its complexity. Thus, aim of this paper is just the solution of equations in question.

1. SOLUTION OF EQUATIONS OF MOTION

1.1 Recapitulation of the problem

For the purposes of better orientation in this problem is in the next page stated Fig. 1, taken over from (1). On the basis of Fig. 1, there are obtained mentioned equations of motion that will be solved in the next subchapter. Designation of variables as well as their purport is identical as in (1).

So, consider the electric hoist (see fig. 1) with mass m_2 [kg] on which the wire rope with length l [m] is suspended. Assuming that wire rope is absolutely rigid and intangible. At the end of wire rope is fitted the load with mass m_4 [kg] and moment of inertia I_{4S} [kg.m^2] relative to the CG of load. Electric hoist is powered by gearmotor with starting torque M [N.m], its moment of inertia including rotating parts reduced to input shaft is I_M [kg.m^2]. Ratio between gear motor and traveling wheels is i [-]. Electric hoist has four traveling wheels and each wheel has a moment of inertia I_3 [kg.m^2] and radius r [m]. Distance from load CG to the suspension point on the fitting is r_S [m].

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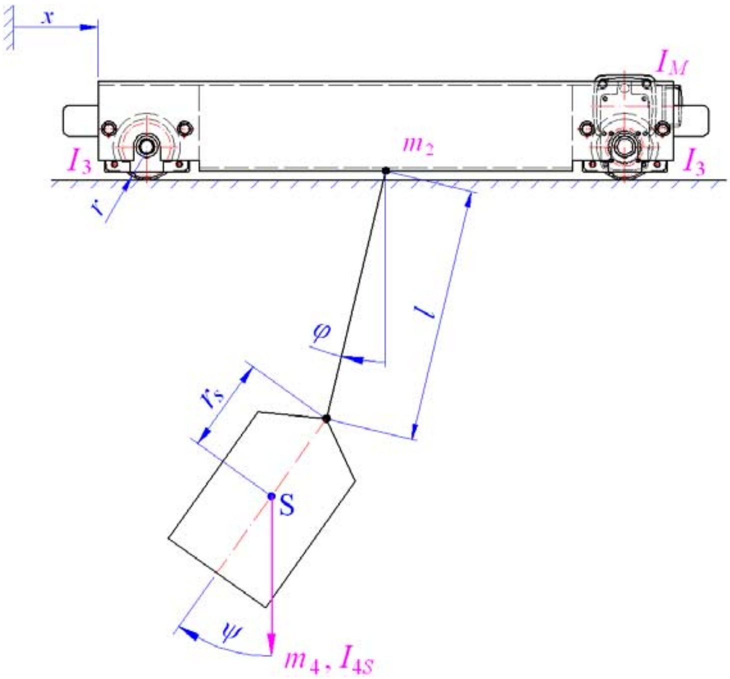


Source: Author, (1)

Fig. 1 – Schematic of the electric hoist with swinging load

Following equations of the motion (after modifications for this solution) were derived by means of Lagrangian mechanics on the basis of Fig. 1.

Equation of motion for coordinate x

$$\ddot{x}\cdot\left(m_2+I_M\cdot\frac{i^2}{r^2}+4\cdot\frac{I_3}{r^2}+m_4\right)-m_4\cdot l\cdot\ddot{\varphi}\cdot\cos(\varphi)+m_4\cdot l\cdot\dot{\varphi}^2\cdot\sin(\varphi)-m_4\cdot r_S\cdot\ddot{\psi}\cdot\cos(\psi)+ \\ +m_4\cdot r_S\cdot\dot{\psi}^2\cdot\sin(\psi)=\frac{M\cdot i}{r} \qquad (1)$$

Equation of motion for coordinate φ

$$l\cdot\ddot{\varphi}-\ddot{x}\cdot\cos(\varphi)+r_S\cdot\ddot{\psi}\cdot\cos(\varphi)\cdot\cos(\psi)-r_S\cdot\dot{\psi}^2\cdot\cos(\varphi)\cdot\sin(\psi)+r_S\cdot\ddot{\psi}\cdot\sin(\varphi)\cdot\sin(\psi)+ \\ +r_S\cdot\dot{\psi}^2\cdot\sin(\varphi)\cdot\cos(\psi)+g\cdot\sin(\varphi)=0 \qquad (2)$$

Equation of motion for coordinate ψ

$$\ddot{\psi}\cdot\frac{\left(I_{4S}+m_4\cdot r_S^2\right)}{m_4\cdot r_S}-\ddot{x}\cdot\cos(\psi)+l\cdot\ddot{\varphi}\cdot\cos(\varphi)\cdot\cos(\psi)-l\cdot\dot{\varphi}^2\cdot\sin(\varphi)\cdot\cos(\psi)+g\cdot\sin(\psi)+ \\ +l\cdot\dot{\varphi}^2\cdot\cos(\varphi)\cdot\sin(\psi)+l\cdot\ddot{\varphi}\cdot\sin(\varphi)\cdot\sin(\psi)=0 \qquad (3)$$

1.2 Implementation of simplifying invariables

For the purposes of simplification of the calculation it is necessary to determine simplifying invariables. These invariables are founded on the basis of electric hoist and load properties, i.e. its mass, moment of inertia, etc.

These constant are stated in the next page in Tab. 1.

Tab. 1 – Properties of an electric hoist and load

Mass of an electric hoist	m_2	163	[kg]
Moment of inertia of the travelling wheel	I_3	0,007859	[kg.m^2]
Travelling wheel radius	r	0,07	[m]
Moment of inertia including rotating part reduced to input shaft	I_M	0,0005	[kg.m^2]
Starting torque of gearmotor	M	10,5	[N.m]
Gear ratio	i	9	[-]
Length of the wire rope	l	2	[m]
Load mass	m_4	100	[kg]
Moment of inertia of the load	I_{4S}	4,687	[kg.m^2]
Distance from load CG to the suspension point on the fitting	r_S	0,38	[m]
Acceleration due to gravity	g	9,81	[m.s^{-2}]

Source: Author

On the basis of Tab. 1 are determined these simplifying invariables

$$A = m_2 + I_M \cdot \frac{i^2}{r^2} + 4 \cdot \frac{I_3}{r^2} + m_4 \tag{4}$$

$$B = m_4 \cdot l \tag{5}$$

$$C = m_4 \cdot r_S \tag{6}$$

$$D = \frac{M \cdot i}{r} \tag{7}$$

$$E = \frac{\left(I_{4S} + m_4 \cdot r_S^2\right)}{m_4 \cdot r_S} \tag{8}$$

After implementation of these invariables into equations (1), (2) and (3) we get

$$A \cdot \ddot{x} - B \cdot \ddot{\varphi} \cdot \cos(\varphi) + B \cdot \dot{\varphi}^2 \cdot \sin(\varphi) - C \cdot \ddot{\psi} \cdot \cos(\psi) + C \cdot \dot{\psi}^2 \cdot \sin(\psi) = D \tag{9}$$

$$\begin{aligned} & l \cdot \ddot{\varphi} - \ddot{x} \cdot \cos(\varphi) + r_S \cdot \ddot{\psi} \cdot \cos(\varphi) \cdot \cos(\psi) - r_S \cdot \dot{\psi}^2 \cdot \cos(\varphi) \cdot \sin(\psi) + r_S \cdot \ddot{\psi} \cdot \sin(\varphi) \cdot \sin(\psi) + \\ & + r_S \cdot \dot{\psi}^2 \cdot \sin(\varphi) \cdot \cos(\psi) + g \cdot \sin(\varphi) = 0 \end{aligned} \tag{10}$$

$$\begin{aligned} & \ddot{\psi} \cdot E - \ddot{x} \cdot \cos(\psi) + l \cdot \ddot{\varphi} \cdot \cos(\varphi) \cdot \cos(\psi) - l \cdot \dot{\varphi}^2 \cdot \sin(\varphi) \cdot \cos(\psi) + g \cdot \sin(\psi) + \\ & + l \cdot \dot{\varphi}^2 \cdot \cos(\varphi) \cdot \sin(\psi) + l \cdot \ddot{\varphi} \cdot \sin(\varphi) \cdot \sin(\psi) = 0 \end{aligned} \tag{11}$$

1.3 Rearranging of equations

Inasmuch as all equations of motion contain coordinate $\ddot{x}$, it is possible to express $\ddot{x}$ (12) from equation (9) and sequentially substitute this obtained equation into equations (10) and (11).

Expressing $\ddot{x}$

$$\ddot{x} = \frac{D + B \cdot \ddot{\varphi} \cdot \cos(\varphi) - B \cdot \dot{\varphi}^2 \cdot \sin(\varphi) + C \cdot \ddot{\psi} \cdot \cos(\psi) - C \cdot \dot{\psi}^2 \cdot \sin(\psi)}{A} \tag{12}$$

And after substituting we get

$$
\begin{aligned}
& l \cdot \ddot{\varphi} - \frac{D + B \cdot \ddot{\varphi} \cdot \cos(\varphi) - B \cdot \dot{\varphi}^2 \cdot \sin(\varphi) + C \cdot \ddot{\psi} \cdot \cos(\psi) - C \cdot \dot{\psi}^2 \cdot \sin(\psi)}{A} \cdot \cos(\varphi) + \\
& + r_S \cdot \ddot{\psi} \cdot \cos(\varphi) \cdot \cos(\psi) - r_S \cdot \dot{\psi}^2 \cdot \cos(\varphi) \cdot \sin(\psi) + r_S \cdot \ddot{\psi} \cdot \sin(\varphi) \cdot \sin(\psi) + \\
& + r_S \cdot \dot{\psi}^2 \cdot \sin(\varphi) \cdot \cos(\psi) + g \cdot \sin(\varphi) = 0
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
& \ddot{\psi} \cdot E - \frac{D + B \cdot \ddot{\varphi} \cdot \cos(\varphi) - B \cdot \dot{\varphi}^2 \cdot \sin(\varphi) + C \cdot \ddot{\psi} \cdot \cos(\psi) - C \cdot \dot{\psi}^2 \cdot \sin(\psi)}{A} \cdot \cos(\psi) + \\
& + l \cdot \ddot{\varphi} \cdot \cos(\varphi) \cdot \cos(\psi) - l \cdot \dot{\varphi}^2 \cdot \sin(\varphi) \cdot \cos(\psi) + g \cdot \sin(\psi) + \\
& + l \cdot \dot{\varphi}^2 \cdot \cos(\varphi) \cdot \sin(\psi) + l \cdot \ddot{\varphi} \cdot \sin(\varphi) \cdot \sin(\psi) = 0
\end{aligned}
\tag{14}
$$

After rearranging of equations (13) and (14) we get

$$
\begin{aligned}
& \ddot{\varphi} \cdot \left[l - \frac{B}{A} \cdot \cos^2(\varphi) \right] + \ddot{\psi} \cdot \left[r_S \cdot \cos(\varphi) \cdot \cos(\psi) + r_S \cdot \sin(\varphi) \cdot \sin(\psi) - \frac{C}{A} \cdot \cos(\psi) \cdot \cos(\varphi) \right] + \\
& \dot{\psi}^2 \cdot \left[\frac{C}{A} \cdot \sin(\psi) \cdot \cos(\varphi) - r_S \cdot \cos(\varphi) \cdot \sin(\psi) + r_S \cdot \sin(\varphi) \cdot \cos(\psi) \right] + \\
& \dot{\varphi}^2 \cdot \frac{B}{A} \cdot \sin(\varphi) \cdot \cos(\varphi) - \frac{D}{A} \cdot \cos(\varphi) + g \cdot \sin(\varphi) = 0
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
& \ddot{\psi} \cdot \left[E - \frac{C}{A} \cdot \cos^2(\psi) \right] + \ddot{\varphi} \cdot \left[l \cdot \sin(\varphi) \cdot \sin(\psi) + l \cdot \cos(\varphi) \cdot \cos(\psi) - \frac{B}{A} \cdot \cos(\varphi) \cdot \cos(\psi) \right] + \\
& \dot{\varphi}^2 \cdot \left[l \cdot \cos(\varphi) \cdot \sin(\psi) + \frac{B}{A} \cdot \sin(\varphi) \cdot \cos(\psi) - l \cdot \sin(\varphi) \cdot \cos(\psi) \right] + \\
& \dot{\psi}^2 \cdot \frac{C}{A} \cdot \sin(\psi) \cdot \cos(\psi) - \frac{D}{A} \cdot \cos(\psi) + g \cdot \sin(\psi) = 0
\end{aligned}
\tag{16}
$$

Equations (15) and (16) above we can simplify by next invariable, thus

$$
F = 1 - \frac{C}{A \cdot r_S} \tag{17}
$$

$$
F = 1 - \frac{B}{A \cdot l} \tag{18}
$$

It is apparent that the magnitude and size of this invariables is identical. After substituting invariable (17) into equation (15) and invariable (18) into equation (16) we get

$$
\begin{aligned}
& \left[l - \frac{B}{A} \cdot \cos^2(\varphi) \right] \cdot \ddot{\varphi} + r_S \cdot [\sin(\varphi) \cdot \sin(\psi) + F \cdot \cos(\varphi) \cdot \cos(\psi)] \cdot \ddot{\psi} = \\
& = -\left[\frac{B}{A} \cdot \sin(\varphi) \cdot \cos(\varphi) \right] \cdot \dot{\varphi}^2 + r_S \cdot [F \cdot \cos(\varphi) \cdot \sin(\psi) - \sin(\varphi) \cdot \cos(\psi)] \cdot \dot{\psi}^2 + \\
& + \frac{D}{A} \cdot \cos(\varphi) - g \cdot \sin(\varphi)
\end{aligned}
\tag{19}
$$

$$l \cdot [\sin(\varphi) \cdot \sin(\psi) + F \cdot \cos(\varphi) \cdot \cos(\psi)] \cdot \ddot{\varphi} + \left[E - \frac{C}{A} \cdot \cos^2(\psi)\right] \cdot \ddot{\psi} =$$

$$= l \cdot [F \cdot \sin(\varphi) \cdot \cos(\psi) - \cos(\varphi) \cdot \sin(\psi)] \cdot \dot{\varphi}^2 - \left[\frac{C}{A} \cdot \sin(\psi) \cdot \cos(\psi)\right] \cdot \dot{\psi}^2 +$$

$$+ \frac{D}{A} \cdot \cos(\psi) - g \cdot \sin(\psi) \tag{20}$$

1.4 Matrix equation

We can equations (19) and (20) express in a matrix notation in order to obtaining a matrix equation.

$$\begin{Bmatrix} l - \frac{B}{A} \cdot \cos^2(\varphi) & r_S \cdot [\sin(\varphi) \cdot \sin(\psi) + F \cdot \cos(\varphi) \cdot \cos(\psi)] \\ l \cdot [\sin(\varphi) \cdot \sin(\psi) + F \cdot \cos(\varphi) \cdot \cos(\psi)] & E - \frac{C}{A} \cdot \cos^2(\psi) \end{Bmatrix} \cdot \begin{Bmatrix} \ddot{\varphi} \\ \ddot{\psi} \end{Bmatrix} =$$

$$\begin{Bmatrix} -\frac{B}{A} \cdot \sin(\varphi) \cdot \cos(\varphi) & r_S \cdot [F \cdot \cos(\varphi) \cdot \sin(\psi) - \sin(\varphi) \cdot \cos(\psi)] \\ l \cdot [F \cdot \sin(\varphi) \cdot \cos(\psi) - \cos(\varphi) \cdot \sin(\psi)] & -\frac{C}{A} \cdot \sin(\psi) \cdot \cos(\psi) \end{Bmatrix} \cdot \begin{Bmatrix} \dot{\varphi}^2 \\ \dot{\psi}^2 \end{Bmatrix} +$$

$$\begin{Bmatrix} \frac{D}{A} \cdot \cos(\varphi) - g \cdot \sin(\varphi) \\ \frac{D}{A} \cdot \cos(\psi) - g \cdot \sin(\psi) \end{Bmatrix} \tag{21}$$

This system will be solved by multiplying inverse matrix to a matrix on the left side. The inverse matrix has a form

$$\frac{\begin{Bmatrix} E - \frac{C}{A} \cdot \cos^2(\psi) & -r_S \cdot [\sin(\varphi) \cdot \sin(\psi) + F \cdot \cos(\varphi) \cdot \cos(\psi)] \\ -l \cdot [\sin(\varphi) \cdot \sin(\psi) + F \cdot \cos(\varphi) \cdot \cos(\psi)] & l - \frac{B}{A} \cdot \cos^2(\varphi) \end{Bmatrix}}{\left[l - \frac{B}{A} \cdot \cos^2(\varphi)\right] \cdot \left[E - \frac{C}{A} \cdot \cos^2(\psi)\right] - l \cdot r_S \cdot [\sin(\varphi) \cdot \sin(\psi) + F \cdot \cos(\varphi) \cdot \cos(\psi)]^2} \tag{22}$$

1.5 Adjustment for Cramer's rule use

If we designate:

$$u_0 = \left[l - \frac{B}{A} \cdot \cos^2(\varphi)\right] \cdot \left[E - \frac{C}{A} \cdot \cos^2(\psi)\right] - l \cdot r_S \cdot [\sin(\varphi) \cdot \sin(\psi) + F \cdot \cos(\varphi) \cdot \cos(\psi)]^2 \tag{23}$$

$$u_{11} = -\left[\frac{B}{A} \cdot \sin(\varphi) \cdot \cos(\varphi)\right] \cdot \left[E - \frac{C}{A} \cdot \cos^2(\psi)\right] - \\ - [\sin(\varphi) \cdot \sin(\psi) + F \cdot \cos(\varphi) \cdot \cos(\psi)] \cdot [F \cdot \sin(\varphi) \cdot \cos(\psi) - \cos(\varphi) \cdot \sin(\psi)] \cdot l \cdot r_S \tag{24}$$

$$u_{12} = \left[E - \frac{C}{A} \cdot \cos^2(\psi)\right] \cdot [F \cdot \cos(\varphi) \cdot \sin(\psi) - \sin(\varphi) \cdot \cos(\psi)] \cdot r_S + \\ + [\sin(\varphi) \cdot \sin(\psi) + F \cdot \cos(\varphi) \cdot \cos(\psi)] \cdot \frac{C}{A} \cdot \sin(\psi) \cdot \cos(\psi) \cdot r_S \tag{25}$$

$$u_{21} = \frac{B}{A} \cdot \sin(\varphi) \cdot \cos(\varphi) \cdot [\sin(\varphi) \cdot \sin(\psi) + F \cdot \cos(\varphi) \cdot \cos(\psi)] \cdot l + \\ + \left[l - \frac{B}{A} \cdot \cos^2(\varphi) \right] \cdot [F \cdot \sin(\varphi) \cdot \cos(\psi) - \cos(\varphi) \cdot \sin(\psi)] \cdot l \tag{26}$$

$$u_{22} = -[\sin(\varphi) \cdot \sin(\psi) + F \cdot \cos(\varphi) \cdot \cos(\psi)] \cdot [F \cdot \cos(\varphi) \cdot \sin(\psi) - \sin(\varphi) \cdot \cos(\psi)] \cdot l \cdot r_S - \\ - \left[l - \frac{B}{A} \cdot \cos^2(\varphi) \right] \cdot \frac{C}{A} \cdot \sin(\psi) \cdot \cos(\psi) \tag{27}$$

$$v_1 = \left[E - \frac{C}{A} \cdot \cos^2(\psi) \right] \cdot \left[\frac{D}{A} \cdot \cos(\varphi) - g \cdot \sin(\varphi) \right] - \\ - [\sin(\varphi) \cdot \sin(\psi) + F \cdot \cos(\varphi) \cdot \cos(\psi)] \cdot \left[\frac{D}{A} \cdot \cos(\psi) - g \cdot \sin(\psi) \right] \cdot r_S \tag{28}$$

$$v_2 = -[\sin(\varphi) \cdot \sin(\psi) + F \cdot \cos(\varphi) \cdot \cos(\psi)] \cdot \left[\frac{D}{A} \cdot \cos(\varphi) - g \cdot \sin(\varphi) \right] \cdot l + \\ + \left[l - \frac{B}{A} \cdot \cos^2(\varphi) \right] \cdot \left[\frac{D}{A} \cdot \cos(\psi) - g \cdot \sin(\psi) \right] \tag{29}$$

Then we obtain from equation (21) on the basis of Cramer's rule a matrix equation (30)

$$\begin{pmatrix} \ddot{\varphi} \\ \ddot{\psi} \end{pmatrix} = \begin{pmatrix} \frac{u_{11}}{u_0} & \frac{u_{12}}{u_0} \\ \frac{u_{21}}{u_0} & \frac{u_{22}}{u_0} \end{pmatrix} \cdot \begin{pmatrix} \dot{\varphi}^2 \\ \dot{\psi}^2 \end{pmatrix} + \begin{pmatrix} \frac{v_1}{u_0} \\ \frac{v_2}{u_0} \end{pmatrix} \tag{30}$$

If we now designate:

$$\varphi_v = \dot{\varphi} \tag{31}$$

$$\psi_v = \dot{\psi} \tag{32}$$

Then is

$$\ddot{\varphi} = \dot{\varphi}_v \tag{33}$$

$$\ddot{\psi} = \dot{\psi}_v \tag{34}$$

Thus, we obtain a system of four equations

$$\dot{\varphi} = \varphi_v \tag{35a}$$

$$\dot{\psi} = \psi_v \tag{35b}$$

$$\dot{\varphi}_v = \frac{u_{11}}{u_0} \cdot \varphi_v^2 + \frac{u_{12}}{u_0} \cdot \psi_v^2 + \frac{v_1}{u_0} \tag{35c}$$

$$\dot{\psi}_v = \frac{u_{21}}{u_0} \cdot \varphi_v^2 + \frac{u_{22}}{u_0} \cdot \psi_v^2 + \frac{v_2}{u_0} \tag{35d}$$

where u_0, u_{11}, u_{12}, u_{21}, u_{22}, v_1, v_2 are the functions of variables φ and ψ

It stands to reason that obtained equations pose a system of four, first-order differential equations. Boundary conditions are

$$\varphi(0) = \psi(0) = \varphi_v(0) = \psi_v(0) = 0, \tag{36}$$

because of stillstand at the beginning.

We can the system described in equations (35a) – (35d) with boundary conditions (36) solve numerically, for instance, by means of two-step method.

If we use a time period ${}_{\Delta}t$ and if we designate:

$t_k = k \cdot {}_{\Delta}t$, $\varphi_k = \varphi(t_k)$, $\psi_k = \psi(t_k)$, $\varphi_{vk} = \varphi_v(t_k)$ and finally $\psi_{vk} = \psi_v(t_k)$, (37)

Then we have

$$\varphi_0 = \psi_0 = \varphi_{v0} = \psi_{v0} = 0 \tag{38}$$

$$\varphi_1 = 0,\ \psi_1 = 0 \tag{39}$$

$$\varphi_{v1} \cong \frac{v_1(0,0)}{u_0(0,0)} \cdot {}_{\Delta}t \quad \text{and} \quad \psi_{v1} \cong \frac{v_2(0,0)}{u_0(0,0)} \cdot {}_{\Delta}t \tag{40}$$

And next

$$\varphi_{k+1} \cong \varphi_{k-1} + 2 \cdot \varphi_{vk} \cdot {}_{\Delta}t \tag{41a}$$

$$\psi_{k+1} \cong \psi_{k-1} + 2 \cdot \psi_{vk} \cdot {}_{\Delta}t \tag{41b}$$

$$\varphi_{vk+1} \cong \varphi_{vk-1} + 2 \cdot {}_{\Delta}t \cdot \left[\frac{u_{11}(\varphi_k,\psi_k)}{u_0(\varphi_k,\psi_k)} \cdot \varphi_{vk}^2 + \frac{u_{12}(\varphi_k,\psi_k)}{u_0(\varphi_k,\psi_k)} \cdot \psi_{vk}^2 + \frac{v_1(\varphi_k,\psi_k)}{u_0(\varphi_k,\psi_k)} \right] \tag{41c}$$

$$\psi_{vk+1} \cong \psi_{vk-1} + 2 \cdot {}_{\Delta}t \cdot \left[\frac{u_{21}(\varphi_k,\psi_k)}{u_0(\varphi_k,\psi_k)} \cdot \varphi_{vk}^2 + \frac{u_{22}(\varphi_k,\psi_k)}{u_0(\varphi_k,\psi_k)} \cdot \psi_{vk}^2 + \frac{v_2(\varphi_k,\psi_k)}{u_0(\varphi_k,\psi_k)} \right] \tag{41d}$$

1.6 Solution of coordinate *x*

For mention, equation (12) has a form

$$\ddot{x} = \frac{D + B \cdot \ddot{\varphi} \cdot \cos(\varphi) - B \cdot \dot{\varphi}^2 \cdot \sin(\varphi) + C \cdot \ddot{\psi} \cdot \cos(\psi) - C \cdot \dot{\psi}^2 \cdot \sin(\psi)}{A}$$

This equation may be rearranged into form

$$\ddot{x} = \frac{D}{A} + \frac{B}{A} \cdot \ddot{\varphi} \cdot \cos(\varphi) - \frac{B}{A} \cdot \dot{\varphi}^2 \cdot \sin(\varphi) + \frac{C}{A} \cdot \ddot{\psi} \cdot \cos(\psi) - \frac{C}{A} \cdot \dot{\psi}^2 \cdot \sin(\psi) \tag{42}$$

After substituting from equation (30) we get

$$\begin{aligned}
\ddot{x} &= \frac{D}{A} - \frac{B}{A} \cdot \dot{\varphi}^2 \cdot \sin(\varphi) - \frac{C}{A} \cdot \dot{\psi}^2 \cdot \sin(\psi) + \frac{B}{A} \cdot \cos(\varphi) \cdot \left(\frac{u_{11} \cdot \dot{\varphi}^2 + u_{12} \cdot \dot{\psi}^2 + v_1}{u_0} \right) + \\
&+ \frac{C}{A} \cdot \cos(\psi) \cdot \left(\frac{u_{21} \cdot \dot{\varphi}^2 + u_{22} \cdot \dot{\psi}^2 + v_2}{u_0} \right) = \\
&= \frac{D}{A} - \frac{B}{A} \cdot \varphi_v^2 \cdot \sin(\varphi) - \frac{C}{A} \cdot \psi_v^2 \cdot \sin(\psi) + \frac{B}{A} \cdot \cos(\varphi) \cdot \frac{u_{11}}{u_0} \cdot \varphi_v^2 + \frac{B}{A} \cdot \cos(\varphi) \cdot \frac{u_{12}}{u_0} \cdot \psi_v^2 + \\
&+ \frac{B}{A} \cdot \cos(\varphi) \cdot \frac{v_1}{u_0} + \frac{C}{A} \cdot \cos(\psi) \cdot \frac{u_{21}}{u_0} \cdot \varphi_v^2 + \frac{C}{A} \cdot \cos(\psi) \cdot \frac{u_{22}}{u_0} \cdot \psi_v^2 + \frac{C}{A} \cdot \cos(\psi) \cdot \frac{v_2}{u_0}
\end{aligned} \tag{43}$$

$$\ddot{x}=\left[-\frac{B}{A}\cdot\sin(\varphi)+\frac{B}{A}\cdot\cos(\varphi)\cdot\frac{u_{11}}{u_0}+\frac{C}{A}\cdot\cos(\psi)\cdot\frac{u_{21}}{u_0}\right]\cdot\varphi_v^2+$$
$$+\left[\frac{B}{A}\cdot\cos(\varphi)\cdot\frac{u_{12}}{u_0}+\frac{C}{A}\cdot\cos(\psi)\cdot\frac{u_{22}}{u_0}-\frac{C}{A}\cdot\sin(\psi)\right]\cdot\psi_v^2+ \tag{44}$$
$$+\frac{D}{A}+\frac{B}{A}\cdot\cos(\varphi)\cdot\frac{v_1}{u_0}+\frac{C}{A}\cdot\cos(\psi)\cdot\frac{v_2}{u_0}$$

Boundary conditions are succeeding

$$x(0)=\dot{x}(0)=0 \tag{45}$$

And if we designate:

$$x_v=\dot{x}\,,\ x_k=x(t_k),\ x_{vk}=x_v(t_k) \tag{46}$$

Then by application of two-step method will be obtained these estimates

$$x_0=x_{v0}=0 \tag{47}$$

$$x_1\cong 0\,,\ x_{v1}\cong\dot{x}_{v0}\cdot{}_{\Delta}t \tag{48}$$

$$x_{k+1}\cong x_{k-1}+2\cdot x_{vk}\cdot{}_{\Delta}t \tag{49}$$

$$x_{vk+1}\cong x_{vk-1}+2\cdot\dot{x}_{vk}\cdot{}_{\Delta}t \tag{50}$$

where

$$\dot{x}_{vk}=\left[-\frac{B}{A}\cdot\sin(\varphi_k)+\frac{B}{A}\cdot\cos(\varphi_k)\cdot\frac{u_{11}(\varphi_k,\psi_k)}{u_0(\varphi_k,\psi_k)}+\frac{C}{A}\cdot\cos(\psi_k)\cdot\frac{u_{21}(\varphi_k,\psi_k)}{u_0(\varphi_k,\psi_k)}\right]\cdot\varphi_{vk}^2+$$
$$+\left[\frac{B}{A}\cdot\cos(\varphi_k)\cdot\frac{u_{12}(\varphi_k,\psi_k)}{u_0(\varphi_k,\psi_k)}+\frac{C}{A}\cdot\cos(\psi_k)\cdot\frac{u_{22}(\varphi_k,\psi_k)}{u_0(\varphi_k,\psi_k)}-\frac{C}{A}\cdot\sin(\psi_k)\right]\cdot\psi_{vk}^2+ \tag{51}$$
$$+\frac{D}{A}+\frac{B}{A}\cdot\cos(\varphi_k)\cdot\frac{v_1(\varphi_k,\psi_k)}{u_0(\varphi_k,\psi_k)}+\frac{C}{A}\cdot\cos(\psi_k)\cdot\frac{v_2(\varphi_k,\psi_k)}{u_0(\varphi_k,\psi_k)}$$

An accuracy of numerical solution depends on time interval magnitude. The smaller is interval ${}_{\Delta}t$, the more accurate is the solution. The error of solution is proportional to square of time interval magnitude $({}_{\Delta}t)^2$.

If the moment of gearmotor is sufficiently small, then will be small the angles φ and ψ as well. For that reason we can write

$$\sin(\varphi)\cong\varphi\,,\ \cos(\varphi)\cong 1 \tag{52}$$

$$\sin(\psi)\cong\psi\,,\ \cos(\psi)\cong 1 \tag{53}$$

Then we can simplify the equations (19) and (20)

$$\left(l-\frac{B}{A}\right)\cdot\ddot{\varphi}+r_S\cdot(\varphi\cdot\psi+F)\cdot\ddot{\psi}\cong-\frac{B}{A}\cdot\varphi\cdot\dot{\varphi}^2+r_S\cdot(F\cdot\psi-\varphi)\cdot\dot{\psi}^2+\frac{D}{A}-g\cdot\varphi \tag{54}$$

$$l\cdot(\varphi\cdot\psi+F)\cdot\ddot{\varphi}+\left(E-\frac{C}{A}\right)\cdot\ddot{\psi}\cong l\cdot(F\cdot\varphi-\psi)\cdot\dot{\varphi}^2-\frac{C}{A}\cdot\psi\cdot\dot{\psi}^2+\frac{D}{A}-g\cdot\psi \tag{55}$$

In addition, because of small magnitude of the angles φ and ψ, it is possible to eliminate all elements in which are these angles present, including their first-order derivatives. Then we get

$$\left(l-\frac{B}{A}\right)\cdot\ddot{\varphi}+r_S\cdot F\cdot\ddot{\psi}\cong\frac{D}{A}-g\cdot\varphi \tag{56}$$

$$l\cdot F\cdot\ddot{\varphi}+\left(E-\frac{C}{A}\right)\cdot\ddot{\psi}\cong\frac{D}{A}-g\cdot\psi \tag{57}$$

We have obtained a system of two linear differential equations with constant coefficients. If we initiate a substitutions

$$\varphi=\frac{D}{A\cdot g}+\alpha \tag{58}$$

$$\psi=\frac{D}{A\cdot g}+\beta \tag{59}$$

We get

$$\begin{pmatrix} l-\frac{B}{A} & r_S\cdot F \\ l\cdot F & E-\frac{C}{A}\end{pmatrix}\cdot\begin{pmatrix}\ddot{\alpha}\\ \ddot{\beta}\end{pmatrix}+g\cdot\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=0 \tag{60}$$

If we multiply second equation (from a matrix notation (60)) by number $p\in R$ and tot up this equation with first equation from a matrix notation, then we get

$$\begin{aligned}&\left(l-\frac{B}{A}+p\cdot l\cdot F\right)\cdot\left(\ddot{\alpha}+p\cdot\ddot{\beta}\right)+g\cdot\left(\alpha+p\cdot\beta\right)+\\&+\left(r_S\cdot F+p\cdot E-p\cdot\frac{C}{A}-p\cdot l+p\cdot\frac{B}{A}-p^2\cdot l\cdot F\right)\cdot\ddot{\beta}=0\end{aligned} \tag{61}$$

$$\begin{aligned}&\left(l-\frac{B}{A}+p\cdot l\cdot F\right)\cdot\left(\ddot{\alpha}+p\cdot\ddot{\beta}\right)+g\cdot\left(\alpha+p\cdot\beta\right)=\\&=\left[p^2\cdot l\cdot F+p\cdot\left(l-E+\frac{C}{A}-\frac{B}{A}\right)-r_S\cdot F\right]\cdot\ddot{\beta}\end{aligned} \tag{62}$$

If we designate p_1, p_2 as the roots of equation (63)

$$p^2\cdot l\cdot F+p\cdot\left(l-E+\frac{C-B}{A}\right)-r_S\cdot F=0 \tag{63}$$

$$p_{1,2}=\frac{-\left(l-E+\frac{C-B}{A}\right)\pm\sqrt{\left(l-E+\frac{C-B}{A}\right)^2-4\cdot l\cdot r_S\cdot F^2}}{2\cdot l\cdot F} \tag{64}$$

Then we get

$$\left(l-\frac{B}{A}+p_k\cdot l\cdot F\right)\cdot\left(\ddot{\alpha}+p_k\cdot\ddot{\beta}\right)+g\cdot\left(\alpha+p_k\cdot\beta\right)=0 \tag{65}$$

If we designate

$$T_k=2\cdot\pi\cdot\sqrt{\frac{l-\frac{B}{A}+p_k\cdot l\cdot F}{g}} \tag{66}$$

$$\gamma_1 = \alpha + p_1 \cdot \beta$$
$$\gamma_2 = \alpha + p_2 \cdot \beta \tag{67}$$

We have

$$\gamma_1 \cong \lambda_{11} \cdot \cos\left(\frac{2 \cdot \pi \cdot t}{T_1}\right) + \lambda_{12} \cdot \sin\left(\frac{2 \cdot \pi \cdot t}{T_1}\right)$$
$$\gamma_2 \cong \lambda_{21} \cdot \cos\left(\frac{2 \cdot \pi \cdot t}{T_2}\right) + \lambda_{22} \cdot \sin\left(\frac{2 \cdot \pi \cdot t}{T_2}\right) \tag{68}$$

After substitution from equations (58) and (59) we get

$$\varphi + p_1 \cdot \psi \cong (1 + p_1) \cdot \frac{D}{A \cdot g} + \lambda_{11} \cdot \cos\left(\frac{2 \cdot \pi \cdot t}{T_1}\right) + \lambda_{12} \cdot \sin\left(\frac{2 \cdot \pi \cdot t}{T_1}\right)$$
$$\varphi + p_2 \cdot \psi \cong (1 + p_2) \cdot \frac{D}{A \cdot g} + \lambda_{21} \cdot \cos\left(\frac{2 \cdot \pi \cdot t}{T_2}\right) + \lambda_{22} \cdot \sin\left(\frac{2 \cdot \pi \cdot t}{T_2}\right) \tag{69}$$

And because of $\varphi(0) = \dot{\varphi}(0) = \psi(0) = \dot{\psi}(0)$, we have

$$\varphi + p_1 \cdot \psi \cong (1 + p_1) \cdot \frac{D}{A \cdot g} \cdot \left[1 - \cos\left(\frac{2 \cdot \pi \cdot t}{T_1}\right)\right] \tag{70}$$

$$\varphi + p_2 \cdot \psi \cong (1 + p_2) \cdot \frac{D}{A \cdot g} \cdot \left[1 - \cos\left(\frac{2 \cdot \pi \cdot t}{T_2}\right)\right] \tag{71}$$

And finally

$$\varphi \cong \frac{D}{A \cdot g} \cdot \left[1 - \frac{p_2 + p_1 \cdot p_2}{p_2 - p_1} \cdot \cos\left(\frac{2 \cdot \pi \cdot t}{T_1}\right) + \frac{p_1 + p_1 \cdot p_2}{p_2 - p_1} \cdot \cos\left(\frac{2 \cdot \pi \cdot t}{T_2}\right)\right] \tag{72}$$

$$\psi \cong \frac{D}{A \cdot g} \cdot \left[1 + \frac{1 + p_1}{p_2 - p_1} \cdot \cos\left(\frac{2 \cdot \pi \cdot t}{T_1}\right) - \frac{1 + p_2}{p_2 - p_1} \cdot \cos\left(\frac{2 \cdot \pi \cdot t}{T_2}\right)\right] \tag{73}$$

Because

$\left|\cos\left(\frac{2 \cdot \pi \cdot t}{T_1}\right)\right| \leq 1$ and $\left|\cos\left(\frac{2 \cdot \pi \cdot t}{T_2}\right)\right| \leq 1$, we have

$$\varphi_{\max} \cong \frac{D}{A \cdot g} \cdot \left[1 + \frac{|p_2 \cdot (1 + p_1)| + |p_1 \cdot (1 + p_2)|}{|p_2 - p_1|}\right] \tag{74}$$

$$\psi_{\max} \cong \frac{D}{A \cdot g} \cdot \left[1 + \frac{|1 + p_1| + |1 + p_2|}{|p_2 - p_1|}\right] \tag{75}$$

From Viète’s formulas is according to equation (64):

$$p_1 \cdot p_2 = -\frac{r_S \cdot F}{l \cdot F} = -\frac{r_S}{l} \quad \Rightarrow \quad -\frac{r_S}{l} < 0 \tag{76}$$

Thus, if we designate roots so that $p_1 < 0,\ p_2 > 0$, (77)

and $f(p) = p^2 \cdot l \cdot F + p \cdot \left(l - E + \frac{C - B}{A}\right) - r_S \cdot F$ (78)

then is $f(0) = -r_S \cdot F < 0$ (79)

$$f(-1) = l \cdot F - l + E - \frac{C}{A} + \frac{B}{A} - r_S \cdot F = l \cdot \left(1 - \frac{C}{A \cdot r_S}\right) - l + E + \frac{B-C}{A} - r_S \cdot \left(1 - \frac{C}{A \cdot r_S}\right) =$$

$$= -\frac{l \cdot C}{A \cdot r_S} + E + \frac{B-C}{A} - r_S + \frac{C}{A} = -\frac{l \cdot C}{A \cdot r_S} + E + \frac{B}{A} - r_S = -\frac{l \cdot B \cdot \frac{r_S}{l}}{A \cdot r_S} + E + \frac{B}{A} - r_S = \quad (80)$$

$$= E - r_S = r_S + \frac{I_{4S}}{m_4 \cdot r_S} - r_S = \frac{I_{4S}}{m_4 \cdot r_S} > 0$$

Necessarily has to be: $-1 < p_1 < 0 < p_2$ (81)

After substitution into equations (74) and (75) we get the maximum values of angles.

$$\varphi_{\max} \cong 2 \cdot \frac{D}{A \cdot g} = 2 \cdot \frac{M \cdot i}{A \cdot r \cdot g} \quad (82)$$

$$\psi_{\max} \cong 2 \cdot \frac{D}{A \cdot g} \cdot \frac{p_2 + 1}{p_2 - p_1} \quad (83)$$

Now, if we want to estimate x, then we have to in equation (42) eliminate small terms $\frac{B}{A} \cdot \dot{\varphi}^2 \cdot \sin(\varphi)$, $\frac{C}{A} \cdot \dot{\psi}^2 \cdot \sin(\psi)$ and if we use $\cos(\varphi) \cong 1$, $\cos(\psi) \cong 1$ whereby we get

$$\ddot{x} \cong \frac{D}{A} + \frac{B}{A} \cdot \ddot{\varphi} + \frac{C}{A} \cdot \ddot{\psi} \quad (84)$$

And after substitution into equations (72) and (73) we get

$$\ddot{x} \cong \frac{D}{A} + \frac{B \cdot D}{A^2 \cdot g} \cdot \left(\frac{2 \cdot \pi}{T_1}\right)^2 \cdot \frac{p_2 + p_1 \cdot p_2}{p_2 - p_1} \cdot \cos\left(\frac{2 \cdot \pi \cdot t}{T_1}\right) -$$
$$- \frac{B \cdot D}{A^2 \cdot g} \cdot \left(\frac{2 \cdot \pi}{T_2}\right)^2 \cdot \frac{p_1 + p_1 \cdot p_2}{p_2 - p_1} \cdot \cos\left(\frac{2 \cdot \pi \cdot t}{T_2}\right) - \frac{C \cdot D}{A^2 \cdot g} \cdot \left(\frac{2 \cdot \pi}{T_1}\right)^2 \cdot \frac{1 + p_1}{p_2 - p_1} \cdot \cos\left(\frac{2 \cdot \pi \cdot t}{T_1}\right) + \quad (85)$$
$$+ \frac{C \cdot D}{A^2 \cdot g} \cdot \left(\frac{2 \cdot \pi}{T_2}\right)^2 \cdot \frac{1 + p_2}{p_2 - p_1} \cdot \cos\left(\frac{2 \cdot \pi \cdot t}{T_2}\right)$$

Thus

$$\ddot{x} \cong \frac{D}{A} + \begin{bmatrix} (B \cdot p_2 - C) \cdot \frac{1 + p_1}{p_2 - p_1} \cdot \left(\frac{2 \cdot \pi}{T_1}\right)^2 \cdot \cos\left(\frac{2 \cdot \pi \cdot t}{T_1}\right) - \\ (B \cdot p_1 - C) \cdot \frac{1 + p_2}{p_2 - p_1} \cdot \left(\frac{2 \cdot \pi}{T_2}\right)^2 \cdot \cos\left(\frac{2 \cdot \pi \cdot t}{T_2}\right) \end{bmatrix} \cdot \frac{D}{A^2 \cdot g} \quad (86)$$

By means of integration will be obtain $\dot{x}$, so that

$$\dot{x} \cong \frac{D}{A} \cdot t + \frac{D}{A^2 \cdot g} \cdot \begin{bmatrix} (B \cdot p_2 - C) \cdot \frac{1 + p_1}{p_2 - p_1} \cdot \left(\frac{2 \cdot \pi}{T_1}\right) \cdot \sin\left(\frac{2 \cdot \pi \cdot t}{T_1}\right) - \\ (B \cdot p_1 - C) \cdot \frac{1 + p_2}{p_2 - p_1} \cdot \left(\frac{2 \cdot \pi}{T_2}\right) \cdot \sin\left(\frac{2 \cdot \pi \cdot t}{T_2}\right) \end{bmatrix} + C_1 \quad (87)$$

After the next integration will be obtained x, so that

$$x \cong \frac{1}{2} \cdot \frac{D}{A} \cdot t^2 - \frac{D}{A^2 \cdot g} \cdot \left(B \cdot p_2 - C\right) \cdot \frac{1+p_1}{p_2 - p_1} \cdot \cos\left(\frac{2 \cdot \pi \cdot t}{T_1}\right) + \\ + \frac{D}{A^2 \cdot g} \cdot \left(B \cdot p_1 - C\right) \cdot \frac{1+p_2}{p_2 - p_1} \cdot \cos\left(\frac{2 \cdot \pi \cdot t}{T_2}\right) + C_1 \cdot t + C_2 \tag{88}$$

Because of $x(0) = \dot{x}(0) = 0$, we obtain from equation (88) solution of constants of integration, so that

$$C_1 = 0 \text{ and } C_2 = \frac{\left(B + C\right) \cdot D}{A^2 \cdot g} \tag{89}$$

And after substitution we get the final form of x

$$x \cong \frac{1}{2} \cdot \frac{D}{A} \cdot t^2 + \frac{\left(B + C\right) \cdot D}{A^2 \cdot g} - \frac{\left(B \cdot p_2 - C\right) \cdot D}{A^2 \cdot g} \cdot \frac{1+p_1}{p_2 - p_1} \cdot \cos\left(\frac{2 \cdot \pi \cdot t}{T_1}\right) + \\ + \frac{\left(B \cdot p_1 - C\right) \cdot D}{A^2 \cdot g} \cdot \frac{1+p_2}{p_2 - p_1} \cdot \cos\left(\frac{2 \cdot \pi \cdot t}{T_2}\right) \tag{90}$$

1.7 Recapitulation of obtained equations

Below are stated the solved equations of motion for coordinates x, φ and ψ. These equations are valid on the assumption that the deflections φ and ψ are sufficiently small.

$$x \cong \frac{1}{2} \cdot \frac{D}{A} \cdot t^2 + \frac{\left(B + C\right) \cdot D}{A^2 \cdot g} - \frac{\left(B \cdot p_2 - C\right) \cdot D}{A^2 \cdot g} \cdot \frac{1+p_1}{p_2 - p_1} \cdot \cos\left(\frac{2 \cdot \pi \cdot t}{T_1}\right) + \\ + \frac{\left(B \cdot p_1 - C\right) \cdot D}{A^2 \cdot g} \cdot \frac{1+p_2}{p_2 - p_1} \cdot \cos\left(\frac{2 \cdot \pi \cdot t}{T_2}\right)$$

$$\varphi \cong \frac{D}{A \cdot g} \cdot \left[1 - \frac{p_2 + p_1 \cdot p_2}{p_2 - p_1} \cdot \cos\left(\frac{2 \cdot \pi \cdot t}{T_1}\right) + \frac{p_1 + p_1 \cdot p_2}{p_2 - p_1} \cdot \cos\left(\frac{2 \cdot \pi \cdot t}{T_2}\right)\right]$$

$$\psi \cong \frac{D}{A \cdot g} \cdot \left[1 + \frac{1+p_1}{p_2 - p_1} \cdot \cos\left(\frac{2 \cdot \pi \cdot t}{T_1}\right) - \frac{1+p_2}{p_2 - p_1} \cdot \cos\left(\frac{2 \cdot \pi \cdot t}{T_2}\right)\right]$$

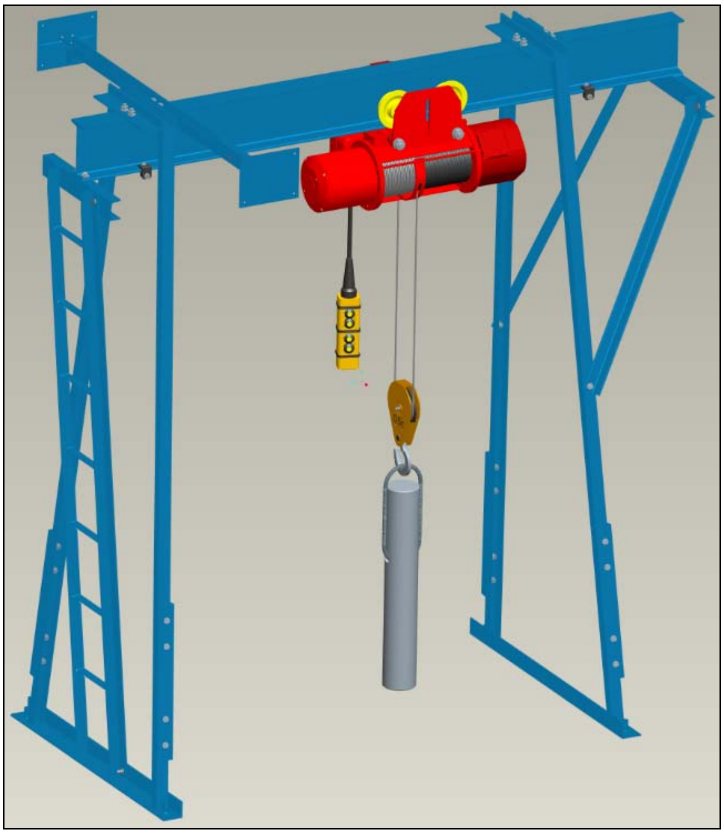
2. GRAPHIC REPRESENTATION OF OBTAINED RELATIONS

On the basis of obtained equations we can carry out graphic representation of coordinates depending upon time and so obtain the first results. To this purpose will be used the values from Tab. 1 that belongs to an actual electric hoist (see Fig. 2).

2.1 Theoretical expectations

Imagine a steel bar with properties (m_4, I_{4S} *and* r_S) according to Tab. 1 suspended on the crane hook. When an electric hoist travels, the load lags behind and sweeps out to a greater height which means increasing potential energy.

It is apparent that in extreme position, where the load acceleration is maximal and load velocity is conversely minimal, the load overshoots the wire rope; i.e. value of angle of the load deflection will be greater than value of angle of the wire rope deflection. Then, as well as at the beginning, the angles compare themselves. Inasmuch as it is not possible to include all kinds of resistances, the process circulates itself and gives large values.

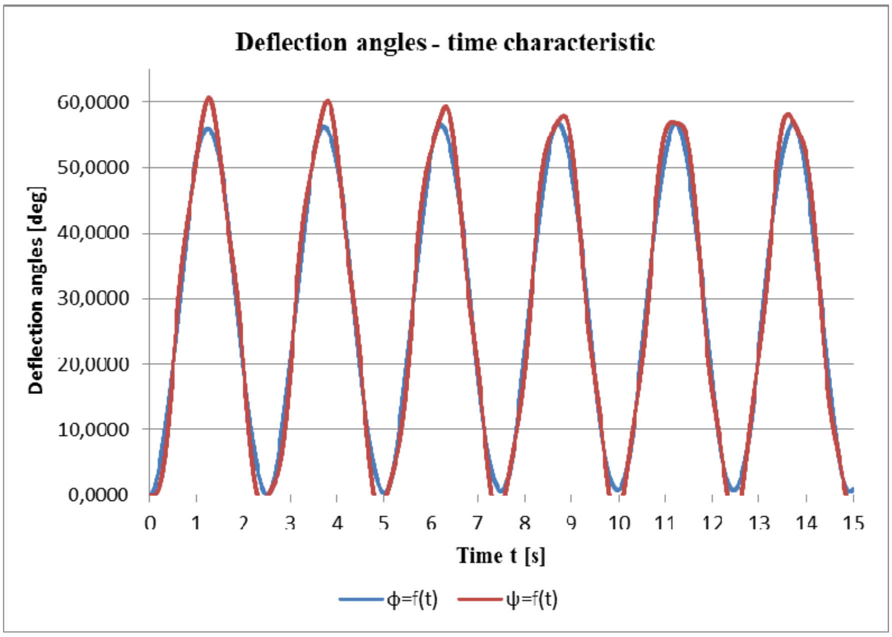


Source: Author

Fig. 2 – 3D model of the measurement assembly with an electric hoist and suspended load

2.2 Graphs

Below we can see the first time relation, i.e. angle φ and angle ψ depending upon time.

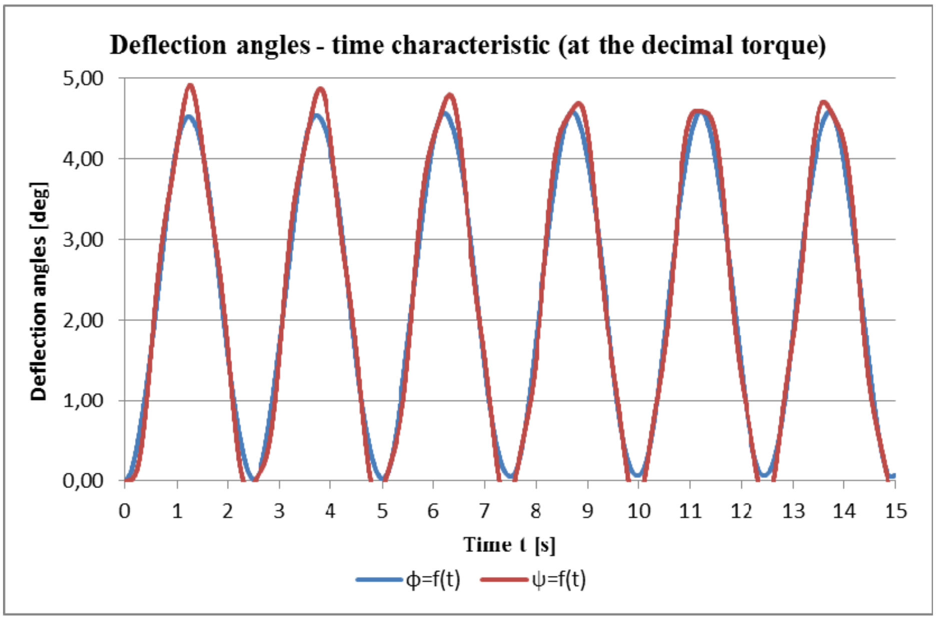


Source: Author

Fig. 3 – Time-characteristic of deflection angles

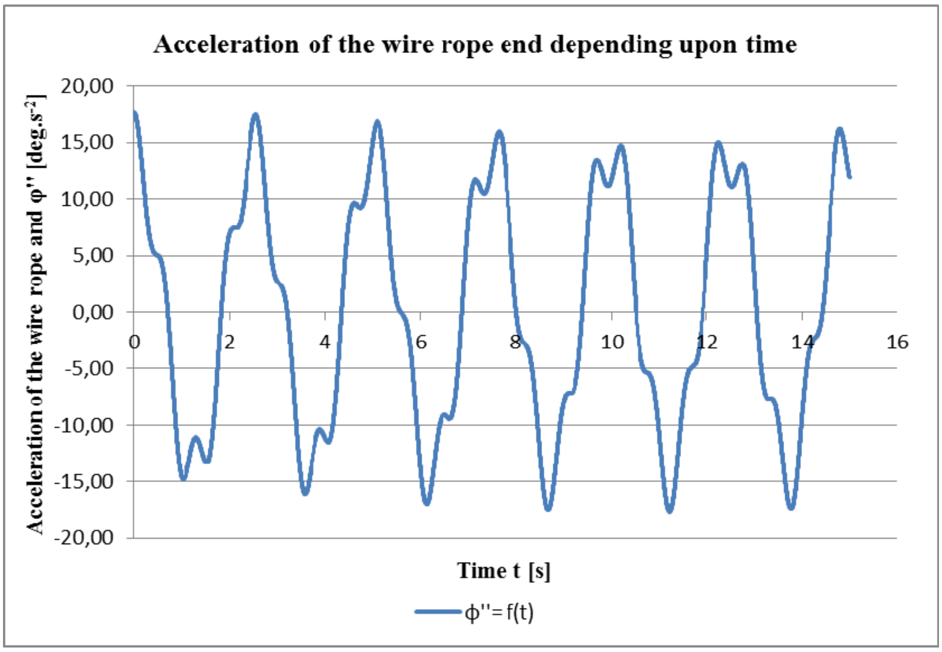
In the Fig. 3 we can obviously see too large values of deflection angles. It is due to resistances which have not been considered. In addition, it is very difficult to include resistance due to wire rope arrangement, particularly its rigidity. Starting torque of gearmotor, concretely its value is not easy to determine as well. For want of values from actual measuring, it's not possible to compare obtained data with real data.

In the event that the starting torque of gearmotor is decimal, then are the time-characteristics more realistic than the last, see Fig. 4.



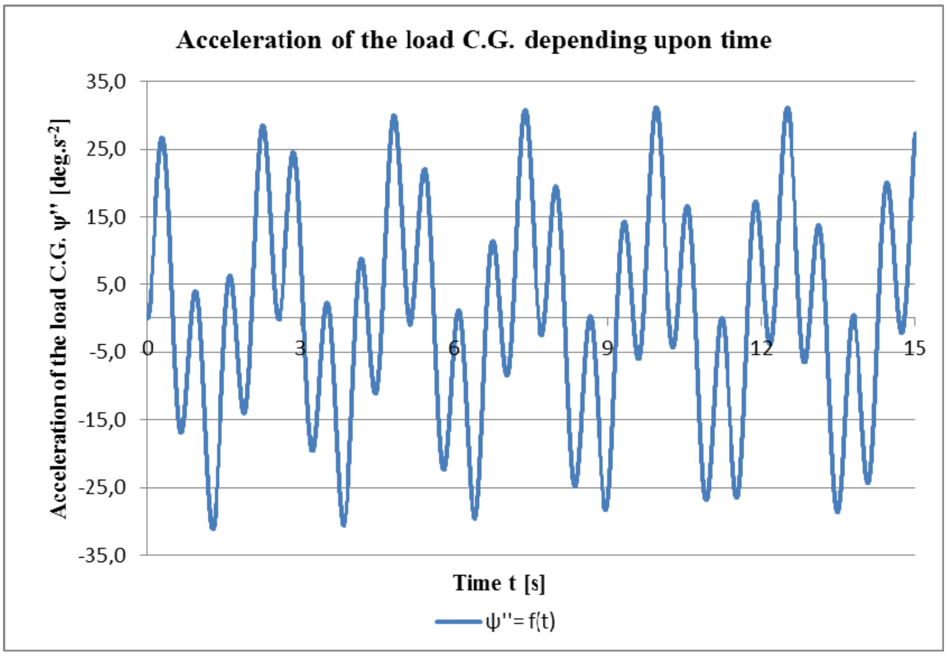
Source: Author

Fig. 4 – Time-characteristic of deflection angles at the decimal starting torque



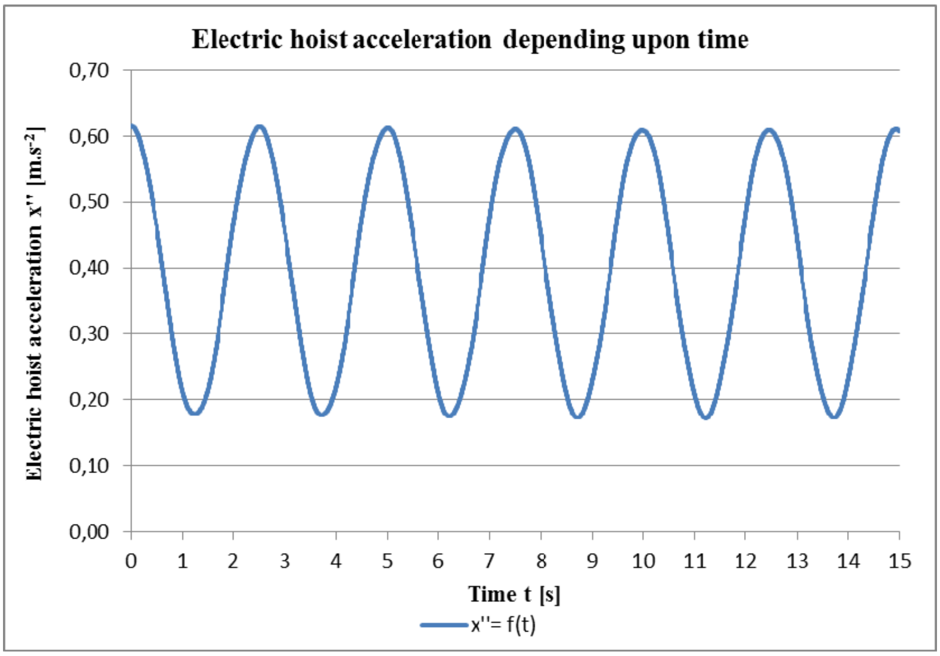
Source: Author

Fig. 5 – Acceleration of the wire rope end depending upon time



Source: Author

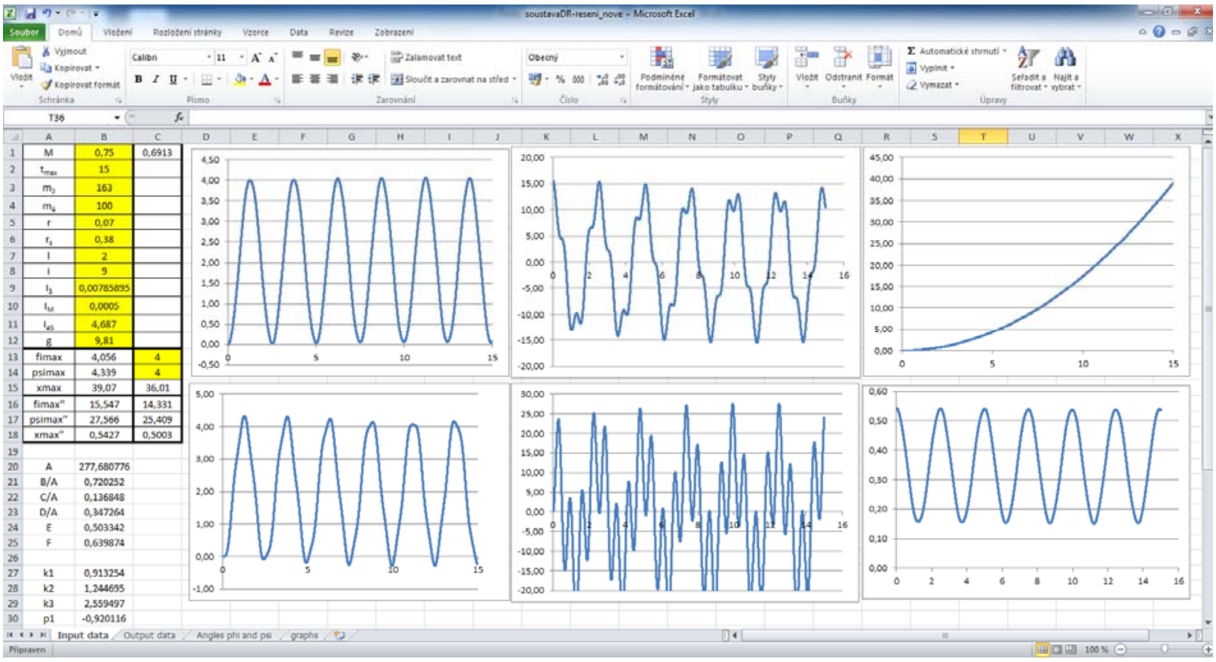
Fig. 6 – Acceleration of the load C.G. depending upon time



Source: Author

Fig. 7 – Electric hoist acceleration depending upon time

All characteristics stated above have been performed by means of an approximate solution in MS Excel 2010 environment (see Fig. 8). This programmed solution makes it possible to change any constant stated in Tab. 1 and according to these constants immediately re-counts the results. Values of mentioned constant are entering into the yellow cells.



Source: Author

Fig. 8 – An approximate solution in MS Excel environment

By means of this approximate solution we can observe the influence of particular constant.

3. CONCLUSION

On the basis of described calculation, the solution of equations of motion for the start of an electric hoist with three degrees of freedom has been performed. Sequentially, according to obtained data, are stated time characteristics of solved variables. These characteristics are the first results of mentioned system at all and give useful information for the next research.

4. ACKNOWLEDGEMENT

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